Searching for S-duality in Gravitation*

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Abstract

We overview some attempts to find S-duality analogues of non-supersymmetric Yang-Mills theory, in the context of gravity theories. The case of MacDowell-Mansouri gauge theory of gravity is discussed. Three-dimensional dimensional reductions from the topological gravitational sector in four dimensions, enable to recuperate the $2+1$ Chern-Simons gravity and the corresponding S-dual theory, from the notion of self-duality in the four-dimensional theory.

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I. INTRODUCTION

Strong/weak coupling duality (S-duality) in superstring and supersymmetric gauge theories in various dimensions has been, in the last five years, the major tool to study the strong coupling dynamics of these theories. Much of these results require supersymmetry through the notion of BPS state. These states describe the physical spectrum and they are protected of quantum corrections leaving the strong/coupling duality under control to extract physical information. In the non-supersymmetric case there are no BPS states and the situation is much more involved. This latter case is an open question and it is still under current investigation.

In the specific case of non-supersymmetric gauge theories in four dimensions, the subject has been explored recently in the Abelian as well as in the non-Abelian cases [1,2] (for a review see [3]). In the Abelian case, one considers CP non-conserving Maxwell theory on a curved compact four-manifold X with Euclidean signature or, in other words, U(1) gauge theory with a θ vacuum coupled to four-dimensional gravity. The manifold X is basically described by its associated classical topological invariants: the Euler characteristic $\chi(X) = \frac{1}{16\pi^2}\int_X \mathrm{tr} R\wedge \tilde{R}$ and the signature $\sigma(X) = -\frac{1}{24\pi^2}\int_X \mathrm{tr} R\wedge R$. In the Maxwell theory, the partition function $Z(\tau)$ transforms as a modular form under a finite index subgroup $\Gamma_0(2)$ of SL$(2,\mathbf{Z})$ [1], $Z(-1/\tau) = \tau^u\bar{\tau}^v Z(\tau)$, with the modular weight $(u,v) = (\frac{1}{4}(\chi+\sigma), \frac{1}{4}(\chi-\sigma))$. In the above formula $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, where g is the U(1) electromagnetic coupling constant and θ is the usual theta angle.

In order to cancel the modular anomaly in Abelian theories, it is known that one has to choose certain holomorphic couplings $B(\tau)$ and $C(\tau)$ in the topological gravitational (non-dynamical) sector, through the action

$$I^{TOP} = \int_X \Big(B(\tau)\mathrm{tr} R\wedge\tilde{R} + C(\tau)\mathrm{tr} R\wedge R\Big), \tag{1}$$

i.e., which is proportional to the appropriate sum of the Euler characteristic $\chi(X)$ and the signature $\sigma(X)$.

II. S-DUALITY IN MACDOWELL-MANSOURI GAUGE THEORY OF GRAVITY

Let us briefly review the MacDowell-Mansouri (MM) proposal [4]. The starting point for the construction of this theory is to consider an SO(3,2) gauge theory with a Lie algebra-valued gauge potential A_μ^{AB}, where the indices $\mu = 0,1,2,3$ are space-time indices and the indices $A, B = 0,1,2,3,4$. From the gauge potential ${A_\mu}^{AB}$ we may introduce the corresponding field strength ${F_{\mu\nu}}^{AB} = \partial_\mu {A_\nu}^{AB} - \partial_\nu A_\mu^{AB} + \frac{1}{2} f_{CDEF}^{AB} A_\mu^{CD} A_\nu^{EF}$, where f_{CDEF}^{AB} are the structure constants of SO(3,2). MM choose $F_{\mu\nu}^{a4} \equiv 0$ and as an action

$$S_{MM} = \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} {F_{\mu\nu}}^{ab} {F_{\alpha\beta}}^{cd}, \tag{2}$$

where $a, b, ...\text{etc.} = 0,1,2,3$.

On the other hand, by considering the self-dual (or anti-self-dual) part of the connection, a generalization has been proposed [5]. The extension to the supergravity case is considered in [6].

One can then search whether the construction of a linear combination of the corresponding self-dual and anti-self-dual parts of the MacDowell-Mansouri action can be reduced to the standard MM action plus a kind of Θ-term and, moreover, if by this means one can find the "dual-theory" associated with the MM theory. This was showed in [7] and the corresponding extension to supergravity is given at [8]. In what follows we follow Ref. [7]. Let us consider the action

$$S = \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} \Big({}^+\tau {}^+{F_{\mu\nu}}^{ab+} {F_{\alpha\beta}}^{cd} - {}^-\tau {}^-{F_{\mu\nu}}^{ab-} {F_{\alpha\beta}}^{cd} \Big), \tag{3}$$

where ${}^\pm {F_{\mu\nu}}^{ab} = \frac{1}{2} \Big({F_{\mu\nu}}^{ab} \pm {\tilde{F}_{\mu\nu}}{}^{ad} \Big)$ and ${\tilde{F}_{\mu\nu}}{}^{ab} = -\frac{1}{2} i \epsilon^{ab}{}_{cd} {F_{\mu\nu}}^{cd}$. It can be easily shown [7], that this action can be rewritten as

$$S = \frac{1}{2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} \Big[({}^+\tau - {}^-\tau) {F_{\mu\nu}}^{ab} {F_{\alpha\beta}}^{cd} + ({}^+\tau + {}^-\tau) {F_{\mu\nu}}^{ab} {\tilde{F}_{\alpha\beta}}{}^{cd} \Big]. \tag{4}$$

In their original paper, MM have shown [4] that the first term in this action reduces to the Euler topological term plus the Einstein-Hilbert action with a cosmological term. This was

achieved after identifying the components of the gauge field A_μ^{AB} with the Ricci rotation coefficients and the vierbein. Similarly, the second term can be shown to be equal to $i\theta P$, where P is the Pontrjagin topological term [5]. Thus, it is a genuine θ term, with θ given by the sum ${}^+\tau + {}^-\tau$.

Our second task is to find the "dual theory", following the same scheme as for Yang-Mills theories [2]. For that purpose we consider the parent action

$$I = \int d^4x \epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\Big(c_1 {}^+G_{\mu\nu}{}^{ab+}G_{\alpha\beta}{}^{cd} + c_2 {}^-G_{\mu\nu}{}^{ab-}G_{\alpha\beta}{}^{cd} + c_3 {}^+F_{\mu\nu}{}^{ab+}G_{\alpha\beta}{}^{cd} + c_4 {}^-F_{\mu\nu}{}^{ab-}G_{\alpha\beta}{}^{cd}\Big). \tag{5}$$

From which the action (3) can be recovered after integration on ${}^+G$ and ${}^-G$.

In order to get the "dual theory" one should start with the partition function

$$Z = \int \mathcal{D}^+G\,\mathcal{D}^-G\,\mathcal{D}A\; e^{-I}. \tag{6}$$

To proceed with the integration over the gauge fields we observe that $F_{\mu\nu}{}^{ab} = \partial_\mu A_\nu{}^{ab} - \partial_\nu A_\mu{}^{ab} + \frac{1}{2} f^{ab}_{CDEF} A_\mu{}^{CD} A_\nu{}^{EF}$. Taking into account the explicit expression for the structure constants, the second term of $F_{\mu\nu}{}^{ab}$ will naturally split in four terms given by $A_\mu{}^{ad} A_{\nu d}{}^{b}$ $-A_\nu{}^{ad} A_{\mu d}{}^{b}$ $-\lambda^2\Big(A_\mu{}^{a4} A_\nu{}^{b4} - A_\nu{}^{a4} A_\mu{}^{b4}\Big)$. The integration over the components $A_\mu{}^{a4}$ is given by a Gaussian integral, which turns out to be $det\,\mathbf{G}^{-1/2}$, where $\mathbf{G}$ is a matrix given by $\mathbf{G}_{ab}{}^{\mu\nu}$ $= 8i\lambda^2\epsilon^{\mu\nu\alpha\beta}\Big(c_3 {}^+G_{\alpha\beta ab} - c_4 {}^-G_{\alpha\beta ab}\Big)$.

Thus, the partition function (6) can be written as

$$Z = \int \mathcal{D}^+G\,\mathcal{D}^-G\,\mathcal{D}A_\mu{}^{ab}\; det\,\mathbf{G}^{-1/2}\; e^{-I\!I}, \tag{7}$$

where

$$I\!I = 2i\int d^4x \epsilon^{\mu\nu\alpha\beta}\Big[c_1 {}^+G_{\mu\nu}{}^{ab+}G_{\alpha\beta ab} - c_2 {}^-G_{\mu\nu}{}^{ab-}G_{\alpha\beta ab} + 2H_{\mu\nu}{}^{ab}(c_3 {}^+G_{\alpha\beta ab} - c_4 {}^-G_{\alpha\beta ab})\Big],$$

and $H_{\mu\nu}{}^{ab} = \partial_\mu A_\nu{}^{ab} - \partial_\nu A_\mu{}^{ab} + \frac{1}{2} f^{ab}_{cdef} A_\mu{}^{cd} A_\nu{}^{ef}$ is the SO(3,1) field strength.

Our last step to get the dual action is to integrate over $A_\mu{}^{ab}$. This kind of integration is well known and has been performed in previous works [2,7,9]. The result is

$$Z = \int \mathcal{D}^+ G\, \mathcal{D}^- G\ det\, \mathbf{G}^{-1/2}\ det(^+M)^{-1/2} det(^-M)^{-1/2}\ e^{-\int d^4x \tilde{L}}, \tag{8}$$

with

$$\tilde{L} = \epsilon^{\mu\nu\rho\sigma}\Big[-\frac{1}{4^+\tau}{}^+G_{\mu\nu}{}^{ab+}G_{\rho\sigma ab} + \frac{1}{4^-\tau}{}^-G_{\mu\nu}{}^{ab-}G_{\rho\sigma ab} + 2\partial_\nu{}^+G_{\rho\sigma ab}(^+M)^{-1\ abcd}_{\mu\lambda}\epsilon^{\lambda\theta\alpha\beta}\partial_\theta{}^+G_{\alpha\beta cd} \\ -2\partial_\nu{}^-G_{\rho\sigma ab}(^-M)^{-1\ abcd}_{\mu\lambda}\epsilon^{\lambda\theta\alpha\beta}\partial_\theta G^-_{\alpha\beta cd}\Big], \tag{9}$$

where ${}^\pm M^{\mu\nu}{}_{ab}{}^{cd} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\Big(-\delta^c_a{}^\pm G_{\alpha\beta b}{}^d + \delta^c_b{}^\pm G_{\alpha\beta a}{}^d + \delta^d_a{}^\pm G_{\alpha\beta b}{}^c - \delta^d_b{}^\pm G_{\alpha\beta a}{}^c\Big)$ and ${}^+\tau = -\frac{1}{4c_1}$, ${}^-\tau = -\frac{1}{4c_2}$, $c_3 = c_4 = 1$.

The non-dynamical model considered in a previous work [9] results in a kind of non-linear sigma model [2] of the type considered by Freedman and Townsend [10], as in the usual Yang-Mills dual models. The dual to the dynamical gravitational model (9) considered here, results in a Lagrangian of the same structure. However, it differs from the non-dynamical case by the features discussed above.

III. (ANTI)SELF-DUALITY OF THE THREE-DIMENSIONAL CHERN-SIMONS GRAVITY

It is well known that the $2+1$ Einstein-Hilbert action with nonvanishing cosmological constant λ is given by the "standard" and "exotic" Einstein actions [11]. It is well known that for $\lambda > 0$ (and $\lambda < 0$), these actions are equivalent to a Chern-Simons actions in $2+1$ dimensions with gauge group $\mathcal{G}$ to be SO(3,1) (and SO(2,2)).

In this section we will work out the Chern-Simons Lagrangian for (anti)self-dual gauge connection with respect to duality transformations of the internal indices of the gauge group $\mathcal{G}$, in the same philosophy of MM [4], and that of [5]

$$L^{\pm}{}_{CS} = \int_{\mathcal{M}} \varepsilon^{ijk}\Big({}^\pm A_i^{AB}\partial_j\ {}^\pm A_{kAB} + \frac{2}{3}{}^\pm A^B_{iA}\,{}^\pm A^C_{jB}\,{}^\pm A^A_{kC}\Big), \tag{10}$$

where $A,B,C,D = 0,1,2,3$, $\eta_{AB} = diag(-1,+1,+1,+1)$ and the complex (anti) self-dual connections are ${}^\pm A_i^{AB} = \frac{1}{2}(A_i^{AB} \mp \frac{i}{2}\varepsilon^{AB}{}_{CD}A_i^{CD})$, which satisfy the relation $\varepsilon^{AB}{}_{CD}{}^\pm A_i^{CD} = \pm i^\pm A^{AB}$.

Thus using the above equations, the action (10) can be rewritten as

$$L^{\pm}{}_{CS} = \int_{\mathcal{M}} \frac{1}{2}\varepsilon^{ijk}\Big(A_i^{AB}\partial_j A_{kAB} + \frac{2}{3}A_{iA}{}^{B}A_{jB}{}^{C}A_{kC}{}^{A}\Big) \mp \frac{i}{4}\varepsilon^{ijk}\varepsilon^{ABCD}\Big(A_{iAB}\partial_j A_{kCD} + \frac{2}{3}A_i{}^{E}{}_{A}A_{jEB}A_{kCD}\Big). \tag{11}$$

In this expression the first term is the Chern-Simons action for the gauge group $\mathcal{G}$, while the second term appears as its corresponding "θ-term". The same result was obtained in $3+1$ dimensions when we considered the (anti)self-dual MM action [7], or the (anti)self-dual $3+1$ pure topological gravitational action [9].

One should remark that the two terms in the action (13) are the Chern-Simons and the corresponding "θ-term" for the gauge group $\mathcal{G}$ under consideration. After imposing the particular identification $A_i^{AB} = (A_i^{ab}, A_i^{3a}) = (\omega_i^{ab}, \sqrt{\lambda}e_i^a)$ and $\omega_i^{ab} = \varepsilon^{abc}\omega_{ic}$, the "exotic" and "standard" actions for the gauge group SO(3,1) are given respectively by

$$L_{CS}{}^{\pm} = \int_X \frac{1}{2}\varepsilon^{ijk}\Big(\omega_i^a(\partial_j\omega_{ka} - \partial_k\omega_{ja}) + \frac{2}{3}\varepsilon_{abc}\omega_i^a\omega_j^b\omega_k^c + \lambda e_i^a(\partial_j e_{ka} - \partial_k e_{ia}) - 2\lambda\varepsilon_{abc}e_i^a e_j^b\omega_k^c\Big)$$

$$\pm i\sqrt{\lambda}\varepsilon^{ijk}\Big(e_i^a(\partial_j\omega_{ka} - \partial_k\omega_{ja}) - \varepsilon_{abc}e_i^a\omega_j^b\omega_k^c + \frac{1}{3}\lambda\varepsilon_{abc}e_i^a e_j^b e_k^c\Big), \tag{12}$$

plus surface terms. It is interesting to note that the above action (12) can be obtained from action (1) (for a suitable choice of $B(\tau)$ and $C(\tau)$) by dimensional reduction from X to its boundary $\mathcal{M} = \partial X$. Thus the "standard" action come from the Euler characteristic $\chi(X)$, while the "exotic" action come from the signature $\sigma(X)$.

IV. CHERN-SIMONS GRAVITY DUAL ACTION IN THREE DIMENSIONS

This section is devoted to show that a "dual" action to the Chern-Simons gravity action can be constructed following [12]. Essentially we will repeat the procedure to find the the "dual" action to MM gauge theory given in Sec. II.

We begin from the original non-Abelian Chern-Simons action given by

$$L = \int_{\mathcal{M}} d^3x \frac{g}{4\pi} \varepsilon^{ijk} A_i^{AB} \Big(\partial_j A_{kAB} + \frac{1}{3} f_{ABCDEF} A_j^{CD} A_k^{EF} \Big). \tag{13}$$

Now, as usual we propose a parent action in order to derive the dual action

$$L_D = \int_{\mathcal{M}} d^3x \varepsilon^{ijk} \Big(aB_i^{AB} H_{jkAB} + bA_i^{AB} G_{jkAB} + cB_i^{AB} G_{jkAB} \Big), \tag{14}$$

where $H_{jkAB} = \partial_j A_{kAB} + \frac{1}{3} f_{ABCDEF} A_j^{CD} A_k^{EF}$ and B_i^{AB} and G_{ij}^{AB} are vector and tensor fields on $\mathcal{M}$. It is a very easy matter to show that the action (13) can be derived from this parent action after integration of G fields

The "dual" action L_D^* can be computed as usually in the Euclidean partition function, by integrating first out with respect to the physical degrees of freedom A_i^{AB}. The resulting action is of the Gaussian type in the variable A and thus, after some computations, it is easy to find the "dual" action

$$L_D^* = \int_{\mathcal{M}} d^3x \varepsilon^{ijk} \Big\{ -\frac{3}{4a} (a\partial_i B_{jAB} + bG_{ijAB}) [\mathbf{R}^{-1}]_{kn}^{ABCD} \varepsilon^{lmn} (a\partial_l B_{mCD} + bG_{lmCD}) + c\alpha_i^{AB} G_{jkAB} \Big\}, \tag{15}$$

where $[\mathbf{R}]$ is given by $[\mathbf{R}]_{ABCD}^{ij} = \varepsilon^{ijk} f_{ABCD}^{EF} B_{kEF}$ whose inverse is defined by $[\mathbf{R}]_{ABCD}^{ij} [\mathbf{R}^{-1}]_{jk}^{CDEF} = \delta_k^i \delta_{AB}^{EF}$.

The partition function is finally given by

$$Z = \int \mathcal{D}G \mathcal{D}B \sqrt{det(\mathbf{M}^{-1})} exp(-L_D^*). \tag{16}$$

In this "dual action" the G field is not dynamical and can be integrated out. The integration of this auxiliary field gives

$$L_D^{**} = \int_{\mathcal{M}} d^3x \frac{4\pi}{g} \varepsilon^{lmn} \Big(B_l^{AB} \partial_m B_{nAB} - \frac{4\pi}{g} f_{ABCDEF} B_l^{AB} B_m^{CD} B_n^{EF} \Big). \tag{17}$$

The fields B cannot be rescaled if we impose "periodicity" conditions on them. Thus, this dual action has inverted coupling with respect to the original one (compare with [13] for the Abelian case).

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REFERENCES

[1] E. Witten, Selecta Mathematica, New Series, **1**, 383 (1995).

[2] O. Ganor and J. Sonnenschein, Int. J. Mod. Phys. A **11** (1996) 5701; N. Mohammedi, hep-th/9507040; Y. Lozano, Phys. Lett. B **364** (1995) 19.

[3] F. Quevedo, Nucl. Phys. Proc. Suppl. **61** A (1998) 23.

[4] S.W. Mac Dowell and F. Mansouri, Phys. Rev. Lett. **38** (1977) 739; F. Mansouri, Phys. Rev. D **16** (1977) 2456.

[5] J.A. Nieto, O. Obregón and J. Socorro, Phys. Rev. D **50** (1994) R3583.

[6] J.A. Nieto, J. Socorro and O. Obregón, Phys. Rev. Lett. **76** (1996) 3482.

[7] H. García-Compeán, O. Obregón and C. Ramírez, Phys. Rev. D **58**, 104012-1 (1998); Chaos, Solitons and Fractals **10** (1999) 373.

[8] H. García-Compeán, J.A. Nieto, O. Obregón and C. Ramírez, Phys. Rev. D **59** (1999) 124003.

[9] H. García-Compeán, O. Obregón, J.F. Plebański and C. Ramírez, Phys. Rev. D **57**, 7501 (1998).

[10] D. Z. Freedman and P. K. Townsend, Nucl. Phys. B **177**, 282 (1981); P. C. West, Phys. Lett. B **76**, 569 (1978); A. H. Chamseddine, Ann. Phys. (N.Y.) **113**, 219 (1978); S. Gotzes and A. C. Hirshfeld, *ibid.* **203**, 410 (1990).

[11] E. Witten, Nucl. Phys. B **311** (1988) 46; Nucl. Phys. B **323** (1989) 113.

[12] H. García-Compeán, O. Obregón, C. Ramírez and M. Sabido, “Remarks on $2+1$ Self-dual Chern-Simons Gravity”, hep-th/9906154, to appear in Phys. Rev. D.

[13] A.P. Balachandran, L. Chandar and B. Sathiapalan, Int. J. Mod. Phys. A **11** (1996).